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NUMERICAL SOLUTION FOR THE STEADY-STATE COEFFICIENTS OF THE INVERSE HEAT-TRANSFER PROBLEM FOR STRATIFIED MEDIA

P. N. Vabishchevich and A. Yu. Denisenko

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Problems of the uniqueness of the inverse heat transfer problem for stratified media are considered and algorithms for computing approximate solutions are discussed.

The coefficients of the inverse problem are of great practical importance in the theory of heat transfer [1, 2]. At present attention is being turned to the problem of determining the thermophysical properties (the coefficients of heat capacity and thermal conductivity), which depend on the temperature. A second important class covers inverse heat transfer problems for stratified media and composite materials. The problem of establishing the temperature dependence of the coefficient of thermal conductivity of a coposite material from temperature measurements within the field has been considered in [3, 4]. In the case of small temperature gradients (small layer thicknesses, large number of layers) it is valid to assume that the thermophysical properties depend on one variable. A steady-state inverse heat transfer problem for a stratified medium is considered in the present paper.

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The problem of the uniqueness of the solution of the inverse problem of the determination of the coefficient of thermal conductivity with respect to additional data at the boundaries of the field is investigated. Various routes are described toward finding approximate solutions by numerical methods, and the case of the piecewise-constant unknown coefficient of thermal conductivity is selected.

Statement of the Problem. The steady-state temperature field in a confined zone D of a stratified medium is described by the equation

$$Lu \equiv -\nabla(k\nabla u) = 0, \quad x = (x_1, x_2) \in D. \tag{1}$$

The coefficient of thermal conductivity k in Eq. (1) depends only on the variable x_1: $k = k(x_1)$. It will be assumed that the heat flux is given at the boundary ∂D of the zone:

$$\frac{\partial u}{\partial n} = \chi(x), \quad x \in \partial D. \tag{2}$$

Without restricting the generality, let us assume that the zone D is convex. The inverse coefficient problem is set up as follows. A pair of functions $\{u(x), k(x_1)\}$ is sought which satisfy Eq. (1) in the zone D and the conditions on the boundary ∂D (the temperature is measured):

$$u(x) = \varphi(x), \quad x \in \partial D. \tag{3}$$

Instead of Eq. (3) the temperature conditions on a part of the boundary $\partial D' \subset \partial D$ may be given.

The inverse problem (1)-(3) arises in the determination of the piecewise-constant coefficient of thermal conductivity $k(x_1)$ of composite materials.

Uniqueness of the Solution of the Inverse Heat Transfer Problem. The usual procedures are used for proving the uniqueness of the solution of the nonlinear inverse problem (1)-(3) [5]. Assume that there are two solutions to the problem (1)-(3), $\{u_1(x), k_1(x_1)\}$ and $\{u_2(x), k_2(x_1)\}$, respectively. Each of them satisfies the equation

$$L_\alpha u_\alpha = 0, \quad x \in D \tag{4}$$

and the boundary conditions

$$u_\alpha(x) = \varphi(x), \quad \frac{\partial u_\alpha}{\partial n}(x) = \chi(x), \quad x \in \partial D, \tag{5}$$

where $\alpha = 1, 2$. For the difference $v(x) = u_1(x) - u_2(x)$ a linear inverse problem is found from (4)-(5) for determining $k_1 - k_2$ from the equation

$$L_1 v = \nabla((k_1 - k_2)\nabla u_2), \quad x \in D, \tag{6}$$

and the boundary conditions

$$v(x) = 0, \quad \frac{\partial v}{\partial n}(x) = 0, \quad x \in \partial D. \tag{7}$$

Allowing for the fact that $k = k(x_1)$, the right-hand part of Eq. (6) is rearranged to the form

$$\nabla((k_1 - k_2)\nabla u_2) = \frac{d(k_1 - k_2)}{dx_1}\frac{\partial u_2}{\partial x_1} + (k_1 - k_2)\Delta u_2. \tag{8}$$

From Eq. (4) with $\alpha = 2$ it is found that

$$k_2 \Delta u_2 = -\frac{dk_2}{dx_1}\frac{\partial u_2}{\partial x_1}. \tag{9}$$

The substitution of (8) and (9) into Eq. (6) gives

$$L_1 v = \frac{\partial u_2}{\partial x_1} k_2 \frac{d}{dx_1}\left(\frac{k_1 - k_2}{k_2}\right). \tag{10}$$

Thus, the uniqueness of the inverse problem (1)-(3) is equivalent to the uniqueness of the solution of the inverse problem (7), (10) by the definition of the right-hand part

$$\eta(x_1) = k_2 \frac{d}{dx_1}\left(\frac{k_1 - k_2}{k_2}\right),$$

which does not depend on the variable x_2. From the form of the right-hand part of $\eta(x_1)$ it follows that the coefficient of thermal conductivity is established apart from a constant multiplier, since with $k_1 = \text{const}\ k_2$, $\eta = 0$. Thus, in order to determine $k(x_1)$ it is necessary to define it at some point $x^* \in D$.

Uniqueness with such additional conditions follows from [6, 7]. In particular, in [7] it is found that $\eta(x_1) = 0$ for a cylindrical region D when $\partial u_2/\partial x_1 \geq 0$ and $\partial^2 u_2/\partial x_1 \partial x_2 \geq 0$. The latter of the restrictions on the behavior of $u_2(x)$ is probably caused by the method of the proof, and no problems occur.

In the case when the additional condition (3) is given on a part of the boundary $\partial D' \subset \partial D$, it is sufficient for uniqueness of the solution that $\partial D'$ is the base of a cylinder [7]. In a rectangular zone $D = \{(x_1, x_2);\ 0 < x_1 < a,\ 0 < x_2 < b\}$ information is given on $\partial D' = \{(x, 0);\ 0 \leq x_1 \leq a\}$.

Numerical Method. For the approximate solution of the problem (1)-(3) the following difference functional is minimized:

$$J(k) = \int_{\partial D} (u(x) - \varphi(x))^2\, dx. \tag{11}$$

Gradient methods [8] are used for this purpose; these are widely applied in the numerical solution of inverse heat transfer problems [1]. The nonstandard element in this approach is connected with the evaluation of the gradient of the functional.

Let us define the gradient J(k) over the increment δk. With an accuracy to terms of the second order of smallness it is then found for the corresponding increment δu that

$$L\delta u + L_\delta u = 0, \quad x \in D, \tag{12}$$

where $L_\delta u \equiv -\nabla(\delta k \nabla u)$. The boundary condition for δu has the form

$$\frac{\partial \delta u}{\partial n}(x) = 0, \quad x \in \partial D. \tag{13}$$

The gradient J(k) is defined on the basis of an investigation of the increment of the Lagrange function [8]:

$$G(k) = J(k) + \int_D \psi(x)\, Lu(x)\, dx.$$

It is found that

$$\delta G = \int_{\partial D} 2(u - \varphi)\,\delta u dx + \int_D \psi(L\delta u + L_\delta u)\, dx. \tag{14}$$

By using Green's formula it is found from (14) that

$$\delta G = \int_{\partial D} 2(u - \varphi)\,\delta u dx + \int_D \delta u L\psi dx + \int_{\partial D} k\left(\psi \frac{\partial \delta u}{\partial n} - \delta u \frac{\partial \psi}{\partial n}\right) dx + \int_D \psi L_\delta u dx. \tag{15}$$

For the last term it is found that

$$\begin{aligned}\int_D \psi L_\delta u dx &= \int_D \nabla(\delta k \psi \nabla u)\, dx + \int_D \delta k \nabla\psi \nabla u dx = \\ &= \int_{\partial D} \delta k \psi \frac{\partial u}{\partial n}\, dx + \int_D \delta k \nabla\psi\nabla u dx = \int_{x_{1\min}}^{x_{1\max}} \delta k(x_1)\, P(x_1)\, dx_1,\end{aligned} \tag{16}$$

where $x_{1\min} = \min_D x_1$, $x_{1\max} = \max_D x_1$, and $P(x_1)$ is defined with respect to $\psi(x)$ and $u(x)$, $x \in D$.

Taking into account the boundary condition (13) and that $\delta G = 0$, it is found from (14)-(16) that

$$\int_{\partial D} 2(u-\varphi)\,\delta u dx + \int_{D} \delta u L\psi dx - \int_{\partial D} k\delta u \frac{\partial \psi}{\partial n}\, dx = \int_{x_{1\min}}^{x_{1\max}} \delta k(x_1)\, P(x_1)\, dx_1. \tag{17}$$

The quantity $\psi(x)$ is determined from the equation

$$L\psi = 0, \quad x \in D, \tag{18}$$

which is supplemented by the boundary condition

$$k\frac{\partial \psi}{\partial n} = 2(u-\varphi), \quad x \in \partial D. \tag{19}$$

In this case the gradient of the functional has the form $\nabla J(k) = P(x_1)$, where $P(x_1)$ is determined according to Eq. (16).

Piecewise-Constant Media. For piecewise constant coefficients $k(x_1)$ the solution has to be sought in the parametric class (finite-difference optimization [8]). In this case $k(x_1)$ is represented in the form

$$k(x_1) = \sum_{i=1}^{m} \alpha_i \eta_i(x_1), \tag{20}$$

where $\eta_i(x_1)$, $i = 1, 2, \ldots, m$, are given and the coefficients of the expansion α_i, $i = 1, 2, \ldots, m$, are determined. For the gradients of the functional (11) it is found when (20) is satisfied that

$$\nabla J(\alpha_1, \alpha_2, \ldots, \alpha_m) = \left\{\frac{\partial J}{\partial \alpha_i}\right\}_{i=1}^{m},$$

where

$$\frac{\partial J}{\partial \alpha_i} = \int_{\partial D} 2(u-\varphi)\,\psi_i dx.$$

The functions $\psi_i(x)$ are found from the solution of the problem

$$L\psi_i = \nabla(\eta_i \nabla u), \quad x \in D, \quad \frac{\partial \psi_i}{\partial n} = 0, \quad x \in \partial D.$$

Difference methods [9] are used in the numerical solution of the boundary value problem for determining $u(x)$, $\psi(x)$, $\psi_i(x)$, $i = 1, 2, \ldots, m$.

The inverse problem (1)-(3) is uncorrected. For establishing a regularized solution use is made of the usual approaches [1, 8] connected with interrupting the iteration process until the functional level of the error is achieved and using a coordinating functional.

A statistical approach is used for establishing the piecewise-constant coefficients $k(x_1)$ with fixed boundaries of the layers. For given limits $\alpha_{i,\min} \le \alpha_i \le \alpha_{i,\max}$, $i = 1, 2, \ldots, \ell$, the statistical selection is carried out on the basis of generating random numbers in an ℓ-dimensional parallelepiped [10]. In practice the limits $\alpha_{i,\min}$, $\alpha_{i,\max}$ are determined from an analysis of a priori information. The algorithm for the statistical selection of the parameters is selected in such a way that the difference between $\alpha_{i,\min}$ and $\alpha_{i,\max}$ is narrowed down in the course of the calculations. Such an approach has a number of advantages especially when the well developed and rapid methods for solving the direct problems of the type (1)-(2) are taken into account. In particular, the method is convenient to use in problems with many criteria when different types of information must be dealt with. In addition, this approach makes it possible to use an interactive (dialog) method of operation on the computer.

NOTATION

u, temperature; k, coefficient of thermal conductivity; $x = (x_1, x_2)$, spatial variables; D, convex region; ∂D, boundary of region D; n, normal; χ, heat flux on boundary of region; φ, temperature on boundary of region; ∂u, ∂k, increments in the temperature and coefficient of thermal conductivity; ψ, solution of conjugate problem.

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